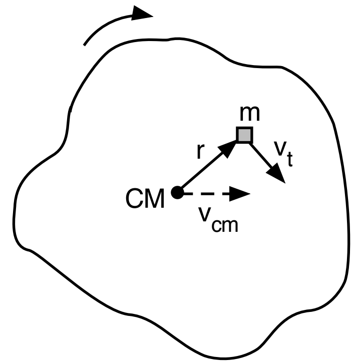
Kinetic Energy

We have stated in class that the kinetic energy of a rigid object that is both moving linearly and rotating about its center of mass is simply the sum of the translational and rotation kinetic energies. Let's prove it.



Imagine an arbitrary object is moving to the right and its center of mass (labeled CM in the diagram) has a velocity of v_{cm}. At the same time, it is also rotating about its center mass, let's say clockwise with an angular velocity of w.

Figuring out the kinetic energy of the object seems straightforward enough: we just break the object into lots of little pieces (one shown in the diagram as m) and add up all the kinetic energies of the little pieces. The annoying part is that all the little pieces have their own individual velocity, v. However, each little velocity can be broken down into two parts: it is moving to the right with velocity v_{cm} because the center of mass is moving and it has a tangential velocity of v_t due to the rotation. So we can say

$$\vec{v} = \vec{v}_{cm} + \vec{v}_t$$

Therefor, the kinetic energy is

$$K = \sum \frac{1}{2}mv^2 = \sum \frac{1}{2}m(\overrightarrow{v_{cm}} + \overrightarrow{v_t})^2 = \sum \frac{1}{2}m(\overrightarrow{v_{cm}}^2 + \overrightarrow{v_t}^2 + 2\overrightarrow{v_{cm}} \cdot \overrightarrow{v_t})$$

Now it is easier to separate that into three separate summations. One will end up being just the translational kinetic energy because the center of mass is moving, another is the rotational kinetic energy because it is rotating about the center of mass, and the third term ends up being zero.

The first term is easy:

$$\sum \frac{1}{2}m\overrightarrow{v_{cm}}^2 = \frac{1}{2}\left(\sum m\right)v_{cm}^2 = \frac{1}{2}mv_{cm}^2$$

For the second term, we have to remember that the tangential velocity for a little mass depends on how far away from the axis it is, v_t=rw. We also have to remember that there is only one rotation rate because it is a rigid body.

$$\sum \frac{1}{2}m\overrightarrow{v_t}^2 = \frac{1}{2}\sum m(r\omega)^2 = \frac{1}{2}\left(\sum mr^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

To show that the last term ends up being zero, we have to remember that when we are multiplying two vectors like that we are taking the dot product of the two vectors. We also need to use v_t=rw, and so the last term becomes

$$\sum \frac{1}{2}m2\overrightarrow{v_{cm}} \cdot \overrightarrow{v_t} = \sum m\overrightarrow{v_{cm}} \cdot \overrightarrow{r\omega}$$

For every single little mass that makes up the object, v_{cm} is a constant, in both magnitude and direction - it always points horizontally. The tangential velocity depends on the little mass. The dot product of two vectors is the projection of one vector onto the second. If two vectors are perpendicular, their dot product is zero. So in this case, the dot product is the velocity of the center of mass times the horizontal component of the tangential velocity. Since v_t=rw, then the horizontal component of the tangential velocity becomes the x-coordinate of the little mass times the rotational speed. That x-coordinate is measure from the center of mass, so we end up with

$$\sum m\overrightarrow{v_{cm}} \cdot \overrightarrow{r\omega} = \sum mv_{cm}x\omega = v_{cm}\omega \sum mx = 0$$

By the definition of center of mass, that last summation is zero! So we can say that the kinetic energy of an object moving and rotating is simply

$$K = K_{translation} + K_{rotation}$$