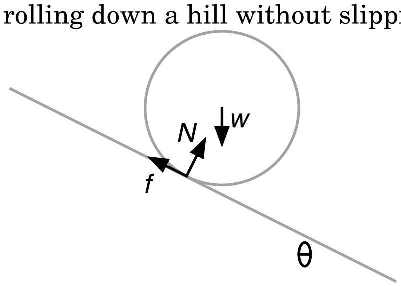
Acceleration Down a Hill

We have seen that the speed of an object that rolls down a hill depends on the shape of the object, but not its mass or radius. It is relatively easy to show that based on energy principles, which is how we have done it so far. It is also possible to conceptually think about why it may be true, based on the idea that a shape with a "larger" moment of inertia will be harder to get rolling so take longer and move slower when rolling down a hill. ("Larger" simply refers to the fraction in the moment of inertia calculation.)

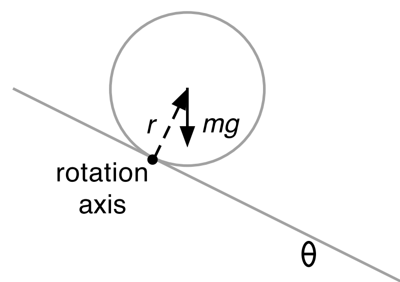
Now let's look at the actual acceleration of a rolling object down a hill by examining the forces and torques acting on a random object rolling down a hill without slipping.



The picture above shows an object on a hill. There are only three forces acting on the object: its weight (acting on the center of mass), friction and normal force (both of which act at the point of contact between the object and the hill.) It is important to note that if the hill is frictionless, it will not roll down the hill! It would simply slide down without rotating.

There are two main ways to analyze the situation, the difference being the choice of rotation axis. We will go through both.

About the Contact Point
Let's begin by examining the net torque on the object based on a rotation axis that is the point of contact with the hill. Though there are three forces, there is only one torque. Friction and the normal force exert no torque about the contact point because they are at the contact point (so "r" is zero for those torque calculations.) This is shown in the diagram to the right.



Applying Newton's Second Law, and calculating the torque exerted by gravity about the contact point we get

$$\sum \tau = I\alpha$$

$$rmg\sin\theta = I\alpha$$

Because we are saying "rolling without slipping", we can say

$$a = r\alpha$$

Now we have to be careful with the moment of inertia. We need the moment of inertia of the object about the contact point with the hill. If we were to look up a moment of inertia, it would be given for an axis about the center of mass. But using the Parallel Axis Theorem we can say

$$I = I_{cm} + mr^2$$

Substituting both of those into Newton's Second Law gives us

$$rmg\sin\theta = \left(I_{cm} + mr^2\right)\left(\frac{a}{r}\right)$$

We then solve for the acceleration to find

$$a = \frac{mr^2 g\sin\theta}{I_{cm} + mr^2}$$

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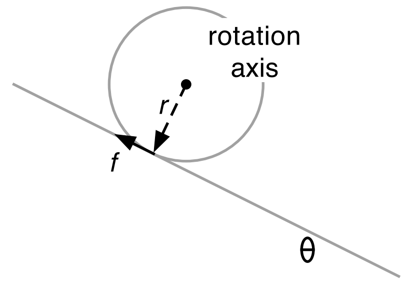
About the Center of Mass

Now let's do Newton's Second Law, but using the center of mass as the rotation point. As before, it turns out that of the three forces acting on the object, only one of them will exert a torque about the center of mass; the force of friction. Gravity acts on the center of mass, so "r" is zero for that torque calculation. The normal force also does not exert a torque, because the normal force is parallel to the vector "r". So Applying Newton's Second Law yields



$$\sum \tau = I\alpha$$

$$rf = I\alpha$$

As before, we know that

$$a = r\alpha$$

This time, the moment of inertia is easy, because our rotation axis is the center of mass.

$$I = I_{cm}$$

In this case, we have to be very careful with the left half of the equation. Remember that we are rolling without slipping, which means that it is a *static* friction acting on the object, so we cannot say that $f = \mu N$. (That expression only gives the maximum possible amount of friction.) The trick is to also apply Newton's Second Law translationally, so that we can say

$$\sum F = ma$$

$$mg\sin\theta - f = ma$$

Solving for the force of friction we get

$$f = mg\sin\theta - ma$$

Now we can substitute all these things into Newton's Second Law to get

$$r\left(mg\sin\theta - ma\right) = I_{cm}\left(\frac{a}{r}\right)$$

Finally, solve the above for the acceleration to finally get

$$a = \frac{mr^2 g\sin\theta}{I_{cm} + mr^2}$$

Notice that we got the same expression as before. While it looks like it depends on the mass and size of the object, it turns out that it does not, as we shall see.

A Slightly More Concrete Example

So what would be the acceleration of a solid sphere down a ramp of base angle ø? Simply look up the moment of inertia for a solid sphere, and plug into our expression derived above:

$$a = \frac{mr^2 g\sin\theta}{\frac{2}{5}mr^2 + mr^2}$$

Which reduces to

$$a = \frac{5}{7}g\sin\theta$$

As (hopefully) expected, the mass and radius have canceled out. The acceleration down the hill only depends on the shape of the object rolling, not its mass or size. Also notice that the acceleration is a little less than if the hill were frictionless.