# The Wave Equation

Let's imagine that we have a long flexible string and that the y-coordinate of the string depends on both the x-coordinate and time as given by the following function y(x,t)

$$y = A\sin(kx + \omega t + \varphi) + B$$

There are five constants in that expression:

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| A | The amplitude of the sine wave |
| k | The wave number, in radians/meter, of the function. The wavelength of the function is given by $\lambda = 2\pi/k$. |
| ω | The angular frequency in radians/second, of the function. The period of the function is given by $T = 2\pi/\omega$. |
| φ | The phase of the function - which basically sets part of the initial conditions. Remember that a cosine and sine are the same function with a different phase. |
| B | The "at rest" height of the string which is the other part of the initial conditions. |

What would be happening to the string? First, imagine we freeze the string at time t = 0. The string would be in the shape of a sine wave with an amplitude of $A$ (shifted horizontally and vertically by the constants $\varphi$ and $B$, which we will now ignore.) The wavelength of the sine wave depends on the value of $k$ by $\lambda = 2\pi/k$. If there was no $\omega t$ term in the function, then nothing would change and the string would be frozen like that. Now imagine we let time advance a little bit. Instead of the function being basically sin (x) it becomes sin (x + "a little bit") - which means the wave would shift to the left a little bit. With each tiny change in time, the wave would shift to the left a little. In other words, we would have a wave that was traveling to the left down the string. If the time term was $-\omega t$ instead, it would move to the right.

So how fast is the wave moving? We know that the period of the wave is $2\pi/\omega$, so that is the time for one wavelength to go by us, which is given by $\lambda = 2\pi/k$. Therefor

$$v = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}$$

Now let's do some math. We will start with our basic function and ignore the initial conditions so that we can say

$$y = A\sin(kx + \omega t)\,.$$

We have a function in both time and space. To see how the function varies in time, we take partial time derivatives:

$$\frac{\partial y}{\partial t} = -A\omega\cos(kx + \omega t)$$

$$\frac{\partial^2 y}{\partial t^2} = -A\omega^2\sin(kx + \omega t) = -\omega^2 y$$

Finally, we can say

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y$$

Notice how similar it simple harmonic motion - the second (partial) time derivative of the function is a negative constant times the function. Now let's do the same thing, but with partial spatial derivatives:

# The Wave Equation

$$\frac{\partial y}{\partial x} = -Ak\cos(kx+\omega t)$$
$$\frac{\partial^2 y}{\partial x^2} = -Ak^2\sin(kx+\omega t) = -k^2 y$$

Again, we can say

$$\frac{\partial^2 y}{\partial x^2} = -k^2 y$$

We have something very similar to before. Notice how we can equate the spatial and temporal derivatives terms by

$$-\frac{1}{k^2}\frac{\partial^2 y}{\partial x^2} = -\frac{1}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

Which we can finally write as

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

Since v = ω/k and is constant, we can say

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| One Dimensional Wave Equation: | $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ |

We call this "one-dimensional" because the wave itself is only traveling in one dimension - along the string. We pictured it as a string vibrating in 2 dimensions - but our function could very easily have described the tension in the string (easier to imagine a slinky) and the string itself would in fact remain flat. This differential equation says the second spatial derivative equals a constant times the second temporal derivative. When that is true, the function represents a wave that moves with a speed that is the inverse square root of the constant.

It turns out that we can generalize this into a wave spreading out in three dimensions by doing partial derivatives in all three dimensions, and we would write that as

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

In this case, our function is called Ψ. This then is usually written in the form

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| Three Dimensional Wave Equation: | $\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$ |