Spacetime Interval

The spacetime interval was introduced by Minkowski with the development of the idea of spacetime. It is crucial to the usefulness of the spacetime conceptualization because it is something that can be calculated, and all reference frames will get the same answer. Worded another way, the spacetime interval is *invariant* for all inertial reference frames.

Let's do a concrete example of an invariant first. In regular cartesian coordinates, the position of two objects would be given by (x_a, y_a, z_a) and (x_b, y_b, z_b). If we used a different set of axes, perhaps with a different origin or rotated, we would have a pair of different coordinates. No matter the coordinates used though, the *distance* between the two points would be the same. If we call the distance between the two points Δs, we could define

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

No matter what set of coordinates we used, we would always calculate the same Δs^2.

Notice how the distance does not involve time at all. This is a "pre-relativity" view of space and time as being two separate concepts. With the introduction of "spacetime," Minkowski pointed out that one needs to explicitly think of time and space somehow merging together in a four-dimensional description of the world. Is there a four-dimensional spacetime analog to the distance between two points? It turns out there is!

Let's consider the following definition of Δs^2 (which we will call the *interval*):

$$(\Delta s)^2 = (\Delta ct)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

Notice how it involves the speed of light and time. If we think about a ray of light moving through space from point A to point B, the interval Δs^2 would be 0. Can you see why? The first term would just be the square of the speed of light multiplied by the time the light moved, so it would be the square of the distance the light traveled. The other three negative terms are also the square of the distance between the two points, which is how far the light traveled.

Here is where it gets interesting, because the fundamental tenant of special relativity is all inertial reference frames measure the same speed of light. Therefore, $\Delta s^2 = 0$ for *every* ray of light in *all* inertial reference frames.

It turns out that the spacetime interval between any 2 events in spacetime is always the same for any inertial reference frame. (I won't prove that here.) The interval is only 0 for rays of light though. When the interval is positive, the two events are said to be *timelike*. All this really means is that it is possible for someone to be present at both events – i.e. they are separated mostly by time. Notice, to be positive, the distance between the two events is less than the distance light would traveled in that time. If the interval is negative, the two events are said to be *spacelike*, meaning they are separated mostly by space. That means it is *not* possible for someone to be present at both events – but the two events could be simultaneous. (And please remember that means the events are only simultaneous in one inertial reference frame and could occur in either order depending on the relative motions.)

In case you are also looking things up on other websites or textbooks, there are actually a few common ways of defining the spacetime interval. Some authors flip the signs and define it as

$$(\Delta s)^2 = -(\Delta ct)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Most people don't write the parentheses either, but I included them to make it clear what is being squared. And as always, if you are using geometric units, there would be no c in the equation, since $c = 1$ in geometric units. Lastly, people often use a matrix notation for this, and some authors will place the space components first, instead of time first as I have done. Just pay attention to the notation used by whatever you are reading or watching.