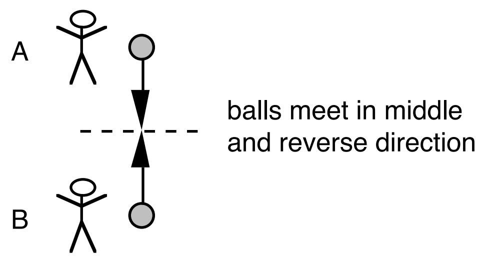
Relativistic Momentum

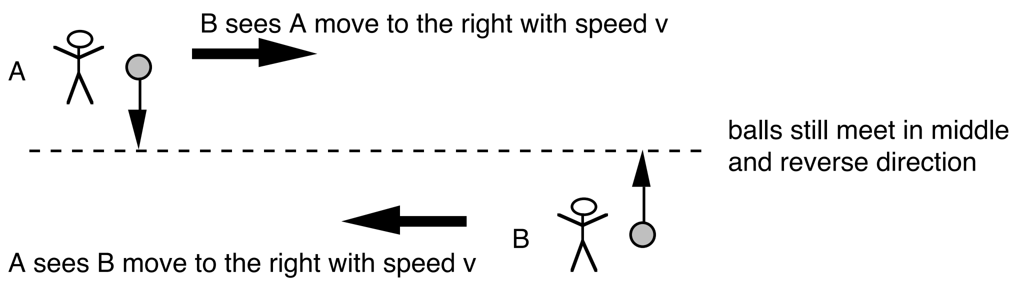
Newtonian mechanics works very well for problems in which speeds are slow compared to light. We spent quite a bit of time solving problems using $\Sigma F = ma$. It should be obvious that this equation cannot hold true when we get to relativistic speeds. If it were valid, then a constant net force would result in a constant acceleration as long as the net force existed; there would be no limit to the speed of the object. It might also seem reasonable that momentum defined as $p = mv$ might have some problems, since it inherently involves the speed of the object. If we are at relativistic speeds, can momentum be conserved?

It turns out that Newton's Second Law is valid, even at relativistic speeds, but only when we use $\Sigma F = dp/dt$ and redefine momentum. Redefining momentum also allows for the more fundamental concept of momentum conservation. To show this, we will look at a strange example, assume momentum is conserved, and see what happens.

Imagine two people are playing with two identical balls. If neither one is moving, and they throw the balls directly at each other with the same speed at the same time, the balls will bounce off each other, reversing their directions. (The total momentum in both cases is zero.) Because there is no relative motion, each person measures and sees the same thing, and they both agree that they threw the balls with the same speed and at the same time.



Imagine now that the two people are moving past each other, each thinking they are at rest and the other moves past them at speed v. They still throw the balls with the same speed they did before, as far as they are concerned, and they time the throws so that the balls still meet in the middle. (Let's call the speed with which they throw the balls u_0, in their own frame of reference.) For momentum to be conserved, the balls will still have to simply bounce and reverse direction.



Now think about the implications because of relativity. A would see B moving in slow motion. In order for the balls to still hit in the middle, A would see B throw her ball first, and then in slow motion, see the ball come toward her. (Obviously, A would still see the ball coming sideways at the speed v.) Why would it be slower? Ball B still has to travel the same distance to meet in the middle (length contraction only occurs in the direction of travel.) but A thinks it takes B a lot longer to travel that distance, according to time dilation. Person B would be seeing the exact same thing of A!

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Calling the distance the ball has to travel d, conservation of momentum would then go like this for person A:

$$m_0 u_0 = mu$$

$$m_0 \frac{d_0}{t_0} = m\frac{d}{t}$$

Since $d_0 = d$ and $t = \gamma t_0$,

$$\frac{m_0}{t_0} = \frac{m}{\gamma t_0}$$

So that

$$m_0 = \frac{1}{\gamma} m$$

And finally,

$$m = \gamma m_0$$

At one point, it was common to talk about relativistic mass, and interpreting the above equation as implying that mass increased as velocity increased. Now, however, it is taken that the mass of an object is always the same, no matter the reference frame, but that momentum is instead defined as

$$p = \gamma mv$$

Defining momentum in this way also makes Newton's Second Law consistent with special relativity, as long as we use the form

$$F = \frac{dp}{dt}$$