Maxwell's Equations & the Speed of Light

Note: Sadly, this handout really requires vector calculus to understand and follow.

In 1865, James Clerk Maxwell published what he called his "Theory of the Electromagnetic Field" that showed how electric and magnetic effects are connected to each other. His ideas were eventually transformed into a set of 4 equations that are now called *Maxwell's Equations*. One can write the equations in integral form or in differential form, both of which are shown below

	Integral form	Differential form	
1.	$\oint \vec{E}\cdot d\vec{l} = -\iint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	a changing magnetic field creates an electric field
2.	$\oint \vec{B}\cdot d\vec{l} = \mu \iint \left(\vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{S}$	$\nabla \times \vec{B} = \mu\left(\vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)$	moving charges or a changing electric field create a magnetic field
3.	$\oiint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon}\iiint \rho \, d\vec{V}$	$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$	a static electric field surrounds a charge
4.	$\oiint \vec{B}\cdot d\vec{S} = 0$	$\nabla \cdot \vec{B} = 0$	"magnetic charges" (a.k.a monopoles) do not exist

E is the electric field, B is the magnetic field and dl, dS and dV are line, surface and volume differentials. J is the current density and ρ is the charge density. μ and ε are terms that relate to how easily magnetic and electric fields spread through a material, respectively. In a vacuum, those terms are called μ_0 and ε_0 and are physical constants in nature: $\mu_0 = 1.257 \times 10^{-6}\ N/A^2$ and $\varepsilon_0 = 8.854 \times 10^{-12}\ F/m$ in SI units.

As part of his 1865 paper, Maxwell calculated how fast an electromagnetic wave would propagate. He discovered that the speed of an electromagnetic was the same as the speed of light at about 3×10^8 m/s, and that an electromagnetic field is polarized, just like light. This led him to conclude that light was an electromagnetic field moving through space.

To determine the speed of an electromagnetic wave, it is much easier to work with the differential forms. We will derive the speed of light through a vacuum - which means we will use μ_0 *and* ε_0 and that there are no random electric charges so that both ρ and J are zero.

Start with the first equation listed above:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

We will take the curl of both sides gives

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

Now, applying the identity in vector calculus for the curl of a curl of a function and remembering that we can do derivatives in any order we have

$$\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\frac{\partial\left(\nabla \times \vec{B}\right)}{\partial t}$$

Since both ρ and J are zero, and using equations 3 and 2, we have

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Which finally becomes

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

That is the wave equation - the differential equation that describes a moving wave!

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We can do the exact same process starting with equation 2. Remembering that J = 0 we would begin with

$$\nabla \times \vec{B} = -\frac{\partial \vec{E}}{\partial t}$$

And we would end up with a similar wave equation

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

In general, the constant in the wave equation is $1/v^2$. The wave equations for both the electric and magnetic fields propagate at the same speed, which we will call "c" for the speed of light. Therefore

$$\frac{1}{c^2} = \mu_0 \varepsilon_0 \rightarrow c = \sqrt{\frac{1}{\mu_0 \varepsilon_0}}$$

The speed of light simply depends on two physical constants in nature, which means there is only one speed at which it can travel. At the end of the nineteenth century, it was largely assumed that light traveled through a special medium that filled the universe, called the aether, and that the speed above was the speed through the aether. However, that assumption also gave an absolute rest frame that violated the principle of relativity. It was this conflict that led Einstein to propose his theory of special relativity.

It should be pointed out that nowadays, the speed of light in a vacuum is taken as the fundamental constant for nature due to special relativity, and μ_0 and ε_0 are defined in terms of the speed of light.