Massless Particles

So far we have derived two new expressions for momentum and kinetic energy: $p = \gamma mv$ and $K = \gamma mc^2 - mc^2$. From the kinetic energy equation, we can define the *rest energy*, E_0, for a particle as $E_0 = mc^2$ and the *total energy* for a particle, E, as $E = \gamma mc^2$, (so that $E = K + E_0$). We have already shown how the equations for momentum and kinetic energy reduce to their classical forms at low speeds. We can combine the equations for momentum and energy and do some algebra to remove the velocity and Lorentz factors and end up with the relationship

$$E^2 = p^2c^2 + m^2c^4$$

To show this we will start by writing out the energy and momentum equations:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$$

Squaring these gives

$$E^2 = \frac{m^2c^4}{1-\frac{v^2}{c^2}} \qquad p^2 = \frac{m^2v^2}{1-\frac{v^2}{c^2}}$$

We can rewrite the momentum equation as

$$p^2c^2 = \frac{m^2c^4\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}$$

Now we subtract the momentum from the energy

$$\begin{aligned}
E^2 - p^2c^2 &= \frac{m^2c^4}{1-\frac{v^2}{c^2}} - \frac{m^2c^4\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} \\
&= \left(\frac{1}{1-\frac{v^2}{c^2}} - \frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right)m^2c^4 \\
&= \left(\frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right)m^2c^4 \\
&= m^2c^4
\end{aligned}$$

Which we can rewrite as

$$E^2 = p^2c^2 + m^2c^4$$

So why do we care about this equation? Well, for starters it shows readily that a particle can have zero mass, yet still have momentum. Photons are particles that have no mass, yet they still have momentum, so that p=E/c. Photons can also only exist at the speed of light. To make this really

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obvious, let's see what happens if we have a particle with a mass of 0. Obviously, from above we can say

$$E = pc$$

but we also know from earlier that the total energy of a particle is given by

$$E = \gamma mc^2$$

Let's equate these and solve for v:

$$pc = \gamma mc^2$$

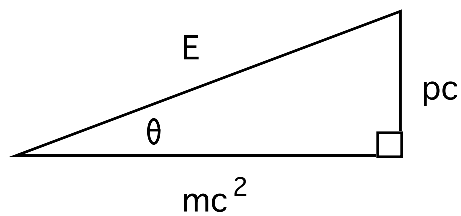
Substitute in the expression for momentum and we get

$$\gamma mvc = \gamma mc^2$$

$$v = c$$

So a massless particle could actually exist, have momentum and energy, but it has to be moving at the speed of light. Since photons can only be measured to be going at the speed of light, that is fine!

There is an interesting visual relating these variables:



Obviously, from the Pythagorean Theorem one can get $E^2 = (pc)^2 + (mc^2)^2$ which we just derived, but it turns out that cosθ and sinθ are interesting also.

For cosθ:

$$\cos\theta = \frac{mc^2}{E} = \frac{mc^2}{\gamma mc^2}$$

$$\cos\theta = \frac{1}{\gamma}$$

For sinθ:

$$\sin\theta = \frac{pc}{E} = \frac{\gamma mvc}{\gamma mc^2} = \frac{v}{c}$$

$$\sin\theta = \beta$$