Lorentz Transform

Background

The curious "failure" of the Michelson-Morley experiment in 1887 to determine the motion of the earth through the aether prompted a lot of physicists to try and figure out why. The first attempt to explain it was in 1889 by the Irish physicist George Fitzgerald, who proposed that perhaps objects shrink the direction of their travel by a little bit due to the distorted electric fields surrounding atoms, causing the intermolecular forces to increase a tiny bit, and thus making an object shrink in the direction of travel. In 1892, the Dutch physicist Hendrik Lorentz proposed a similar idea, and included an effect on how time is measured. Lorentz based his ideas on a mathematical transformation of coordinates due to motion. Later, he also showed that

In June of 1905, the French physicist Henri Poincare expanded the ideas of Lorentz, giving Lorentz the credit for the idea and published what are now known as the Lorentz transformations. He also showed that these transformation equations could be thought of as a rotation in a 4 dimensional space, with the 4th dimension being imaginary and connected to the speed of light and time.

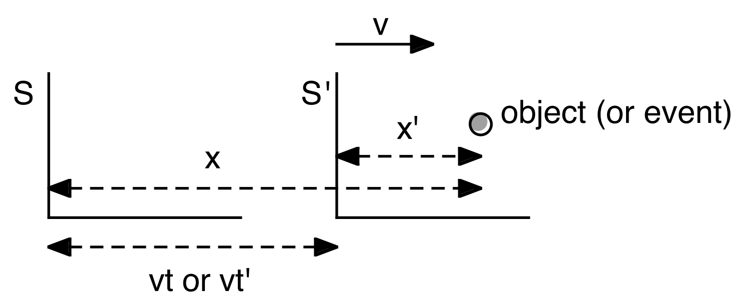
While the equations developed by Lorentz and Poincare were identical to those developed by Einstein, their interpretations of the equations were very different. Before Einstein, it was thought that an object would shrink in their direction of travel because the atoms themselves got a little bit closer in the direction of travel due to the increased electrical forces caused by the motion. The atoms themselves stayed the same. And even though time appeared be transformed by the motion, it was thought of as a mere mathematical trick; the "real" time was the rest frame. It was Einstein, after deriving the transformation equations from his Principle of Relativity, who interpreted the equations as saying that space and time itself were affected by relative motion.

Before we work on deriving the Lorentz transformations, let's first look at the classical Galilean transformation.

Galilean Transform

Galileo was the first person to recognize the idea of inertia – he realized that one couldn't tell if one was moving with a constant velocity or at rest. Newton later fully explained the ideas in his laws of motion. Central to Newton's Laws is the lack of an absolute rest frame and that there is really no difference between being at rest and moving with constant velocity.

Under classical Newtonian mechanics, it is relatively easy to relate positions and times between two inertial reference frames; the hardest thing is keeping track of all the notation. Using standard notation, let's call the two reference frames S and S', and assume they are moving with a constant speed v with respect to each other. That means that S "sees" S' moving to the right at speed v and S' "sees" S moving to the left at speed v. We will always use primes to differentiate between reference frames. Let's keep things a little less complicated by also deciding that the axis of the two reference frames are aligned at t=0; so x=x'=0 when t=t'=0. Now lets look at the diagram below to relate the position of an object or event between the two reference frames.



Lorentz Transform

The variable x is the position of the object in the stationary reference frame S and x' is the position of the same object in the moving reference frame S'. Since S' is moving to the right with a constant velocity of v, it will have a moved a distance of vt in a time of t, so we can write the two equations:

$$x' = x - vt$$

$$x = x' + vt$$

These equations are known as the Galilean Transform between the two reference frames. (We could have talked about this when we did relative motion, but we didn't.) In classical Newtonian mechanics, these equations are correct.

After Maxwell came out with his four equations for electricity and magnetism and showed that an electromagnetic wave can only propagate at one speed, people assumed that meant an electromagnetic wave traveled through the ether with that speed. However, that seemed to introduce the idea of a preferred frame of reference, namely the reference frame of the ether. In the late nineteenth century, a series of incredibly accurate experiments by Michelson (later joined by Morley) attempted and failed to determine the motion of the earth through the ether by looking at light pulses traveling perpendicular to each other and comparing their times of travel. Not only was there the initial philosophical problem of absolute rest (which people had rejected since the time of Galileo), there was now the problem that they couldn't find the absolute rest! In attempting to explain the "negative" results of the Michelson-Morley experiment, a number of people came up with explanations based on a form of time dilation or length contractions. People also recognized that Maxwell's equations for electricity and magnetism failed under the familiar and "obvious" Galilean transformations.

We looked at frames of reference in a simple manner early in the year when we talked about relative motion. At the time, we only really looked at how velocities and accelerations differ between two reference frames moving with respect to each other. We have also hinted at the idea of coordinate transformations when we did things like occasionally decide that "down" was positive to make gravity a positive acceleration, or make the edge of a cliff height "0" with the base of the cliff being a negative height. We chose a convenient coordinate system to describe the problem. The idea of a coordinate transformation is really just an extension of what we did earlier.

Lorentz Transform

Because we already know about time dilation and the issues of simultaneity, we know that we cannot assume that both reference frames measure time in the same way. Now, we will use t' to be time in the S' frame of reference. In addition, we will assume that our equations are similar to the Galilean transforms, but differ by some value, which we shall call "k" as follows:

$$x' = k(x - vt)$$

$$x = k(x' + vt')$$

Now the question is what is the value of k? Thankfully, it is pretty easy to figure out with just a little algebra. We just go back to the most important idea – Maxwell's equations imply that light can only travel with one speed. So let's imagine a light flash that starts at the origin at time t=0. (That also means it starts at t'=0 and at the origin in S', because that is what we said from the beginning.) So both frames must see the light pulse travel at the same speed, so after a certain amount of time (t or t') the light pulse will have traveled a distance x or x' by the following:

$$x' = ct'$$

$$x = ct$$

Lorentz Transform

(Notice that we haven't used the phrase "proper time" or "proper length." These are just position and time coordinates in two reference frames.)

Therefore we can substitute into the previous equations and say

$$ct' = k(ct - vt)$$

$$ct = k(ct' + vt')$$

These can be rewritten as

$$t' = k(1 - \frac{v}{c})t$$

$$t = k(1 + \frac{v}{c})t'$$

To make it easier to write, we will say ß=v/c, and substitute our expression for t into the expression for t', getting

$$t' = k(1 - \beta)k(1 + \beta)t'$$

Which we can finally solve for k as follows

$$1 = k^2(1 - \beta^2)$$

$$k = \frac{1}{\sqrt{1 - \beta^2}}$$

This is the γ term that we have been using all along! This is also why that term is called the Lorentz factor – it was first derived by Lorentz about ten years before Einstein. Now we can finally write the correct transformation equations as

$$x' = \gamma(x - vt)$$

$$x = \gamma(x' + vt')$$

We are not done, however. In the classical Galilean transformation, time is measured the same in both reference frames (so we would say that t'=t.). That is not true in the Lorentz transform. We need to find the transformation equations for time. All it will take is a little algebra: substitute one of the above equations into the other, and then solve for either t or t'.

$$x' = \gamma(x - vt) = \gamma(\gamma(x' + vt') - vt)$$

Now we just have to simplify and solve for t:

$$x' = \gamma^2 x' + \gamma^2 vt' - \gamma vt$$

$$(1 - \gamma^2)x' = \gamma^2 vt' - \gamma vt$$

$$\gamma vt = \gamma^2 vt' - (1 - \gamma^2)x'$$

$$t = \gamma t' - \frac{(1 - \gamma^2)x'}{\gamma v}$$

It turns out it is easier to think about what happens to the $(1 - \gamma^2)/\gamma$ term, and then sub that back into the above relationship.

Lorentz Transform

$$\frac{(1-\gamma^2)}{\gamma} = \frac{\left(1-\frac{1}{1-\beta^2}\right)}{\gamma} = \frac{\left(\frac{1-\beta^2-1}{1-\beta^2}\right)}{\gamma} = \frac{\left(\frac{-\beta^2}{1-\beta^2}\right)}{\gamma}$$

$$= \left(\frac{-\beta^2}{1-\beta^2}\right)\sqrt{1-\beta^2} = \frac{-\beta^2}{\sqrt{1-\beta^2}}$$

$$\frac{(1-\gamma^2)}{\gamma} = -\beta^2\gamma$$

Plugging that back into the previous equation for time we get

$$t = \gamma t' - \frac{(1-\gamma^2)x'}{\gamma v} = \gamma t' - \left(-\beta^2\gamma\right)\frac{x'}{v}$$

$$= \gamma t' + \beta^2\gamma\frac{x'}{v} = \gamma\left(t' + \beta^2\frac{x'}{v}\right)$$

Which finally turns into

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

And we find the reverse equation by switching the primes and reversing the sign of the velocity (as with the position equations)

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

To summarize, here are the final Lorentz transformations:

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

It can be useful to think of these in the context of displacements in time and space, and so to speed up some derivations, we will also write them as:

$$\Delta x' = \gamma(\Delta x - v\Delta t) \qquad \Delta x = \gamma(\Delta x' + v\Delta t')$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \qquad \Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)$$

Lorentz Transform

Consequences of the Lorentz Transform

Simultaneity

From the above equations, it becomes clear that space and the speed of light are connected to time intervals. Think about two events that happen at two different locations, but are simultaneous in the S' frame. (This means that Δt' = 0 and Δx' ≠ 0.) From the transform equation, we see that in the S frame there is a non-zero time interval between the two events

$$\Delta t = \gamma\left(0 + \frac{v\Delta x'}{c^2}\right) = \gamma\frac{v\Delta x'}{c^2}$$

This means that the events are not simultaneous in S.

Time Dilation

We can derive the equations for time dilation pretty easily. Let's say that two events happen at the same location in S'. Therefor, the time interval Δt' is the proper time, and Δx' = 0. What is the time interval in S?

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) = \gamma\left(\Delta t' + \frac{v(0)}{c^2}\right) = \gamma\Delta t'$$

$$t = \gamma t_0$$

Length Contraction

We can also derive the length contraction equation. Imagine an object is at rest in the S' frame and so it's length in the S' frame is L_0, which we could also call Δx'. To measure its length in the S frame (Δx) we would have to measure the positions of its end points at the same time in the S frame, so that Δt = 0. (Note that is a different way of thinking about it than our previous derivation.) Then we can say

$$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma\Delta x$$

$$\Delta x = \frac{1}{\gamma}\Delta x'$$

$$L = \frac{1}{\gamma}L_0$$

Velocity Addition

This is an equation that we did not derive before, as you really need to use these transformations to do it. The question is this: if S' measures an object to have a speed of u', what is the speed of the object in S? By definition, we can say

$$u' = \frac{\Delta x'}{\Delta t'} \text{ and } u = \frac{\Delta x}{\Delta t}$$

and so all we have to do is use the transformations, so that

$$u = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta x' + v\Delta t')}{\gamma(\Delta t' + \frac{v\Delta x'}{c^2})}$$

Dividing through by Δt', this becomes

Lorentz Transform

$$u = \frac{\Delta x' + v\Delta t'}{\Delta t' + \frac{v\Delta x'}{c^2}}\left(\frac{\frac{1}{\Delta t'}}{\frac{1}{\Delta t'}}\right) = \frac{\Delta x'/\Delta t' + v\Delta t'/\Delta t'}{\Delta t'/\Delta t' + v\Delta x'/c^2\Delta t'} = \frac{\Delta x'/\Delta t' + v}{1 + v\Delta x'/c^2\Delta t'}$$

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Flipping the primes and changing the sign of the velocity then gives the inverse transform:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

Let's do an example with this equation, since you haven't used it before. Recall the question:

> *You are watching two spaceships with velocities (relative to you) as shown. You then see ship A fire a laser (which is a beam of light) at ship B.*

> a. *How fast does each ship think the laser pulse is traveling?*
> b. *How fast does ship A think ship B is traveling? (You do not need a number answer, but you can give bounds of the correct answer.)*

Let's answer question b for real.

Ship A will be reference frame S. You are reference frame S'. Both reference frames are looking at ship B. We will use the following equation:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

The following variables are defined as follows:
u = the velocity of B with respect to A, which is what we are trying to find.
u' = the velocity of B with respect to you = –0.8c
v = the velocity of S' with respect to S = –0.5c
c = the speed of light, which we will call 1.

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{-0.8 - 0.5}{1 + \frac{(-0.8)(-0.5)}{1^2}} = \frac{-1.3}{1.4}$$

$$u = -0.928$$

So the spaceships see each other approach at 0.928c, or 2.79 x 10^8 m/s.