Elastic Collisions

Energy and momentum are always conserved in a collision, no matter what happens. Momentum is easy to deal with because there is only "one form" of momentum, (**p**=m**v**), but you do have to remember that momentum is a vector. Energy is tricky because it has many forms, the most troublesome being heat, but also sound and light. If kinetic energy is conserved in a collision, it is called an elastic collision. In an elastic collision, the total kinetic energy is conserved because the objects in question "bounce perfectly" like an ideal elastic. An inelastic collision is one where some of the of the total kinetic energy is transformed into other forms of energy, such as sound and heat. Any collision in which the shapes of the objects are permanently altered, some kinetic energy is always lost to this deformation, and the collision is not elastic. It is common to refer to a "completely inelastic" collision whenever the two objects remain stuck together, but this does not mean that all the kinetic energy is lost; if the objects are still moving, they will still have some kinetic energy.

General Equation Derivation: Elastic Collision in One Dimension

Given two objects, m_1 and m_2, with initial velocities of v_{1i} and v_{2i}, respectively, how fast will they be going after they undergo a completely elastic collision? We can derive some expressions for v_{1f} and v_{2f} by using the conservation of kinetic energy and the conservation of momentum, and a lot of algebra.

Begin by making the following conservation statements:

Conservation of Kinetic Energy: $\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$

Conservation of Momentum: $m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$

To solve for v_{1f} and v_{2f} (which is really two equations in two unknowns), we need some algebra tricks to simplify the substitutions. Take both equations and group them according to the masses: put all the m_1's on one side of the equation and all the m_2's on the other. We'll also cancel out all the 1/2's at this point.

Conservation of Kinetic Energy becomes: $m_1v_{1i}^2 - m_1v_{1f}^2 = m_2v_{2f}^2 - m_2v_{2i}^2$

which can be simplified as $m_1\left(v_{1i}^2 - v_{1f}^2\right) = m_2\left(v_{2f}^2 - v_{2i}^2\right)$ **eqn. 1**

Conservation of Momentum becomes: $m_1v_{1i} - m_1v_{1f} = m_2v_{2f} - m_2v_{2i}$

which can be simplified as $m_1\left(v_{1i} - v_{1f}\right) = m_2\left(v_{2f} - v_{2i}\right)$ **eqn. 2**

Now comes the algebra fun. Divide equation 1 by equation 2.

$$\frac{m_1\left(v_{1i}^2 - v_{1f}^2\right)}{m_1\left(v_{1i} - v_{1f}\right)} = \frac{m_2\left(v_{2f}^2 - v_{2i}^2\right)}{m_2\left(v_{2f} - v_{2i}\right)}$$

After all the cancellations, we are left with: $v_{1i} + v_{1f} = v_{2f} + v_{2i}$ **eqn. 3**

Solving for v_{1f} we get: $v_{1f} = v_{2f} + v_{2i} - v_{1i}$ **eqn. 4**

Now we take equation 4 and substitute back into one of our original equations to solve for v_{2f}. Since the momentum equation is easier, lets use that.

Conservation of Momentum becomes: $m_1v_{1i} + m_2v_{2i} = m_1(v_{2f} + v_{2i} - v_{1i}) + m_2v_{2f}$

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Now do some algebra...

$$m_2 v_{2f} + m_1 v_{2f} = m_1 v_{1i} + m_2 v_{2i} - m_1 \left(v_{2i} - v_{1i} \right)$$

$$\left(m_2 + m_1 \right) v_{2f} = m_1 v_{1i} + m_1 v_{1i} + m_2 v_{2i} - m_1 v_{2i}$$

$$v_{2f} = \frac{\left(m_1 + m_1 \right)}{\left(m_2 + m_1 \right)} v_{1i} + \frac{\left(m_2 - m_1 \right)}{\left(m_2 + m_1 \right)} v_{2i}$$

Until we get: $v_{2f} = \frac{2m_1}{\left(m_2 + m_1 \right)} v_{1i} + \frac{\left(m_2 - m_1 \right)}{\left(m_2 + m_1 \right)} v_{2i}$

Now we substitute this result back into equation 4 do some algebra to solve for v_{1f}.

Equation 4 becomes:

$$v_{1f} = \left[\frac{2m_1}{\left(m_2 + m_1 \right)} v_{1i} + \frac{\left(m_2 - m_1 \right)}{\left(m_2 + m_1 \right)} v_{2i} \right] + v_{2i} - v_{1i}$$

Now do some algebra....

$$v_{1f} = \left[\frac{2m_1}{\left(m_2 + m_1 \right)} - 1 \right] v_{1i} + \left[\frac{\left(m_2 - m_1 \right)}{\left(m_2 + m_1 \right)} + 1 \right] v_{2i}$$

$$v_{1f} = \left[\frac{2m_1 - \left(m_2 + m_1 \right)}{\left(m_2 + m_1 \right)} \right] v_{1i} + \left[\frac{\left(m_2 - m_1 \right) + \left(m_2 + m_1 \right)}{\left(m_2 + m_1 \right)} \right] v_{2i}$$

Until we get: $v_{1f} = \frac{m_1 - m_2}{\left(m_2 + m_1 \right)} v_{1i} + \frac{2m_2}{\left(m_2 + m_1 \right)} v_{2i}$

Questions:

1. What would happen if an object were to have a completely elastic collision with an identical object initially at rest?

2. What would happen if an object were to have a completely elastic collision with an identical object *not* initially at rest?

3. What would happen if a really small object were to collide with a really massive object initially at rest? (i.e. $m_2 >> m_1$)

4. What would happen if a really massive object were to collide with a really small object at initially at rest? (i.e. $m_2 << m_1$)

5. Imagine holding a really light object on top of a really massive object, and then dropping both of them at the same time onto the ground. If all the collisions are elastic, and the objects are dropped from a height of h, how high will the little object bounce?

6. In the previous question, what should be the ratio of the masses if the bottom mass had no velocity after all the collisions? How high would the little mass bounce?