Energy of Elliptical Orbit

Hopefully, you can show that the total energy of a circular orbit is given by the expression

$$E = -\frac{GmM}{2R}$$

Recall that you simply said that the total energy is the sum of the planet's kinetic and potential energies. To handle the "v" in the kinetic energy, you said the centripetal force on the planet was the force of gravity on the planet. A little algebra later, you had derived the above. That equation is valid even for an elliptical orbit, where R is the semi-major axis. To show that, we will follow a similar process, but instead of invoking centripetal force we will use the conservation of angular momentum, and a lot more algebra.

First, we imagine a planet of mass m orbiting a star of mass M in an elliptical orbit. The perihelion distance we will call p and the aphelion distance a. We will also say that the energy of the planet has to be conserved, so that its energy at aphelion is the same as its energy at perihelion, or

$$\frac{1}{2}mv_a^2 - \frac{GmM}{a} = \frac{1}{2}mv_p^2 - \frac{GmM}{p}$$

We also know that the angular momentum of the planet as it goes around the sun is conserved, (and realize that the velocity at perihelion and aphelion is perpendicular to the line connecting the sun and the planet) so that

$$amv_a = pmv_p$$

So we can say

$$v_a = \frac{p}{a}v_p$$

Now substitute this into the energy equation earlier, and solve for v_p.

$$\left(\frac{p}{a}v_p\right)^2 - \frac{2GM}{a} = v_p^2 - \frac{2GM}{p}$$

$$v_p^2\left(\frac{p^2}{a^2} - 1\right) = 2GM\left(\frac{1}{a} - \frac{1}{p}\right)$$

$$v_p^2 = 2GM\left(\frac{1}{a} - \frac{1}{p}\right)\left(\frac{a^2}{p^2 - a^2}\right)$$

$$v_p^2 = 2GM\left(\frac{a}{p(p+a)}\right)$$

To finally get our energy equation, we simply go back to our original statement of the energy of the orbit is the kinetic plus the potential, but now we apply this to the perihelion position, and use our freshly derived expression for the velocity of the planet at perihelion.

$$E = \frac{1}{2}mv_p^2 - G\frac{mM}{p} = \frac{1}{2}m\left(2GM\left(\frac{a}{p(p+a)}\right)\right) - G\frac{mM}{p}$$

No we just have to clean it up a bit as follows:

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$$E = GmM\left(\frac{a}{p(p+a)}\right) - G\frac{mM}{p}$$

$$= GmM\left(\frac{a}{p(p+a)} - \frac{1}{p}\right)$$

$$= GmM\left(\frac{a-(p+a)}{p(p+a)}\right)$$

$$= -GmM\left(\frac{1}{p+a}\right)$$

The last step is to realize that "a+p" is the major axis, 2R, so are finally left with the energy for an elliptical orbit

$$E = -\frac{GmM}{2R}$$