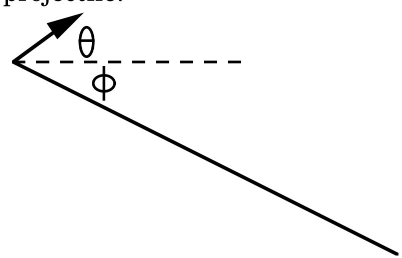
Projectile on a Hill

We have looked at the simple case of a projectile fired across level ground, and you have already derived the expression $R = \frac{v^2}{g}\sin 2\theta$ for the horizontal range of a projectile fired with an initial velocity of v at an angle of θ above the horizontal across a flat level field. From that, we showed the angle that will give the maximum range for a given initial speed was 45°. Now let's do a more complicated example.

A projectile is fired with an initial velocity of v at an angle of θ above the horizontal from the side of a hill that is sloping down at a constant angle of ϕ, as shown in the diagram below. Find an expression for the range down the hill of the projectile, and then find an expression for the maximum range of the projectile.



The initial velocity would be vcosθ**i** + vsinθ**j**. Let's call the initial position of the projectile the origin, so that the and x and y coordinates of the projectile can be written as:

$$x = (v\cos\theta)t$$

$$y = -\tfrac{1}{2}gt^2 + (v\sin\theta)t$$

As the hill is sloping down, the x and y coordinates of where the projectile lands have the relationship (calling ϕ a positive number)

$$y = -x\tan\phi$$

Combining the two expressions for y gives us

$$-x\tan\phi = -\tfrac{1}{2}gt^2 + (v\sin\theta)t$$

To get rid of time in this expression, use the expression $t = x/v\cos\theta$ to get

$$x\tan\phi = \tfrac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2 - (v\sin\theta)\left(\frac{x}{v\cos\theta}\right)$$

Now just do a lot of algebra:

$$\tan\phi = \frac{gx}{2v^2\cos^2\theta} - \frac{\sin\theta}{\cos\theta}$$

$$\frac{gx}{2v^2\cos^2\theta} = \tan\phi + \frac{\sin\theta}{\cos\theta}$$

$$x = \frac{2v^2}{g}\left(\tan\phi\cos^2\theta + \sin\theta\cos\theta\right)$$

To finally get

$$x = \frac{v^2}{g}\left(2\tan\phi\cos^2\theta + \sin 2\theta\right)$$

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To find the angle that gives the maximum range, take the derivative and set it equal to zero:

$$\frac{dx}{d\theta} = \frac{v^2}{g}\left(4\tan\phi\cos\theta(-\sin\theta) + 2\cos 2\theta\right)$$
$$0 = \frac{2v^2}{g}(-\tan\phi\sin 2\theta + \cos 2\theta)$$

So that

$$0 = -\tan\phi\sin 2\theta + \cos 2\theta$$
$$\tan\phi\sin 2\theta = \cos 2\theta$$

Which finally gives

$$\tan 2\theta = \frac{1}{\tan\phi}$$

This will be true if

$$2\theta = 90 - \phi$$

Giving

$$\theta = \frac{90-\phi}{2}$$

As a quick check, we know that if $\phi = 0^\circ$, then this is a horizontal field, and using the above gives us $\theta = (90-0)/2 = 45^\circ$, which we already know to be the correct answer.

In addition, we can also use this to check our expression for x-coordinate we found earlier:

$$x = \frac{v^2}{g}\left(2\tan\phi\cos^2\theta + \sin 2\theta\right)$$

Substituting in $\phi = 0^\circ$, we get

$$x = \frac{v^2}{g}\left(2\tan 0\cos^2\theta + \sin 2\theta\right)$$
$$= \frac{v^2}{g}\sin 2\theta$$

Which is what we had derived in an earlier class for the range of a projectile across a horizontal field.