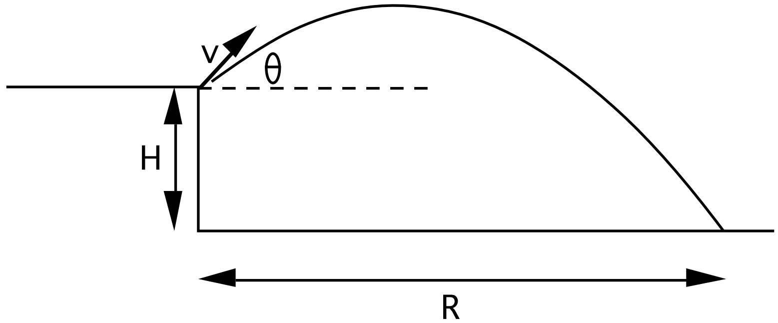
# Projectile off a Cliff

The most fun derivation is when a projectile is fired off a cliff of height H and velocity v at angle θ. What is the angle that will give the maximum range?




The x and y coordinates of the projectile, with an initial height of H, and initial velocity of v @ θ are

$$x = (v\cos\theta)t$$

$$y = -\frac{1}{2}gt^2 + (v\sin\theta)t + H$$

(This would use +9.8 m/s$^2$ for the acceleration due to gravity.)

One method for solving this is to use the expression for y to find an expression for the time when the projectile lands on the ground. Then use this time to find an expression for the horizontal position of the projectile when it hits the ground. Find the maximum value for this range expression by taking the derivative, and you have found the maximum range and the angle that gives that maximum range. Unfortunately, this way leads to a lot of algebra, so we will take a different approach below.

In general, for a given initial speed, there are two initial angles that will cause the projectile to land in the same place. However, at the maximum range, there is only one initial angle. We will use this idea after setting up some algebra.

Solving the x equation for t, and substituting that into the y equation yields the following:

$$y = -\frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2 + (v\sin\theta)\left(\frac{x}{v\cos\theta}\right) + H$$

Simplifying a little bit, and saying that when it hits the ground, y=0 and x=R, gives:

$$0 = -\frac{1}{2}\frac{gR^2}{v^2\cos^2\theta} + R\tan\theta + H$$

Using the trig identity

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$

gives the following expression

$$0 = -\frac{gR^2}{2v^2}\tan^2\theta + R\tan\theta + \left(H - \frac{gR^2}{2v^2}\right)$$

We have a quadratic relationship, and can find the angles θ that solve the above with the quadratic equation. Therefore

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$$\tan\theta = \frac{-R \pm \sqrt{R^2 - 4\left(\frac{-gR^2}{2v^2}\right)\left(H - \frac{gR^2}{2v^2}\right)}}{2\left(\frac{-gR^2}{2v^2}\right)}$$

In general, this doesn't look too nice, but since we are interested in the maximum range and its associated angle, we remember that at the maximum range, there is only one angle. That means the discrimanent must be equal to zero so that

$$0 = R_{max}{}^2 - 4\left(\frac{-gR_{max}{}^2}{2v^2}\right)\left(H - \frac{gR_{max}{}^2}{2v^2}\right)$$

Now solve for $R_{max}$

$$0 = \frac{v^2}{2g} + H - \frac{gR_{max}{}^2}{2v^2}$$

$$\frac{gR_{max}{}^2}{2v^2} = \frac{v^2}{2g} + H$$

$$R_{max}{}^2 = \frac{2v^2}{g}\left(\frac{v^2}{2g} + H\right)$$

$$R_{max}{}^2 = \frac{v^2}{g^2}\left(v^2 + 2gH\right)$$

Until we get the following for the maximum range of the projectile:

$$R_{max} = \frac{v}{g}\sqrt{\left(v^2 + 2gH\right)}$$

To find the angle for the maximum range, plug this into the quadratic equation that started this:

$$\tan\theta = \frac{-R \pm \sqrt{R^2 - 4\left(\frac{-gR^2}{2v^2}\right)\left(H - \frac{gR^2}{2v^2}\right)}}{2\left(\frac{-gR^2}{2v^2}\right)} = \frac{-\left(\frac{v}{g}\sqrt{v^2 + 2gH}\right) \pm 0}{\frac{-g}{v^2}\left(\frac{v^2}{g^2}\left(v^2 + 2gH\right)\right)}$$

Simplifying this gives the following expression for the maximum range angle:

$$\tan\theta = \frac{v}{\sqrt{v^2 + 2gH}}$$

*This was adapted from Bace, Ilijic, and Narancic, "Maximizing the Range of a Projectile,"* *__European Journal of Physics, 23__* *(2002) p. 409-11*

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Brute Force Method:
I won't give all the intermediate steps, but here is the brute force way to solve this, if you want to try and check your work. Be very careful of your algebra; it is very easy to make stupid little mistakes. (I lost count of the number of times I had to do this till I got it right.)

As before, we can write the following for the x and y coordinates of the projectile:

$$x = (v\cos\theta)t$$

$$y = -\frac{1}{2}gt^2 + (v\sin\theta)t + H$$

Therefore the time when the projectile hits the ground is when y=0, so solving for the time and taking the positive root gives the following for the time of impact:

$$t = \frac{v\sin\theta + \sqrt{v^2\sin^2\theta + 2gH}}{g}$$

We can therefore write the following as an expression for the range of the projectile, as a function of the initial angle:

$$R = v\cos\theta\left(\frac{v\sin\theta + \sqrt{v^2\sin^2\theta + 2gH}}{g}\right)$$

To find the maximum range, take the derivative and set it equal to zero. (This is where it gets ugly.)

$$\frac{dR}{d\theta} = v\left(\cos^2\theta - \sin^2\theta\right) + \frac{v^2\sin\theta\cos^2\theta}{\sqrt{v^2\sin^2\theta + 2gH}} - \sin\theta\sqrt{v^2\sin^2\theta + 2gH} = 0$$

Multiplying everything by the square root term, isolating the resulting root, and then squaring everything gives

$$v^2(1 - 2\sin^2\theta)^2(v^2\sin^2\theta + 2gH) = ((v^2 - 2gH)\sin\theta - 2v^2\sin^3\theta)^2$$

In expanding this mess, it turns out that all the higher powers of sine cancel. After a little simplifying, we are left with

$$\sin\theta = \frac{v}{\sqrt{2v^2 + 2gH}}$$

(This actually is the same as our previous answer – can you see why?)