To Kepler from Newton

In the early 17th century, Kepler discovered the following three laws of planetary motion:

1. The planets orbit around the sun in an ellipse with the sun at one focus.
2. As the planets orbit around the sun, they sweep out equal areas from a line drawn to the sun in equal times.
3. The ratio of the period squared to semi-major axis cubed is the same for all the planets orbiting the sun.

Kepler discovered these laws empirically. While he searched for a reason behind the laws, he was never able to discover why the laws were true. He only new that they worked. It wasn't until Newton formulated his three laws of motion and law of gravity that the "why" behind Kepler's laws was found. Newton was the first person to be able to derive Kepler's laws from fundamental principles: his Three Laws of Motion and his Theory of Universal Gravitation.

Imagine that there is a planet of mass m orbiting a sun of mass M. Putting the sun at the origin, the net force on the planet would simply be the force of gravity between the planet and the sun, so we can write Newton's Second Law as

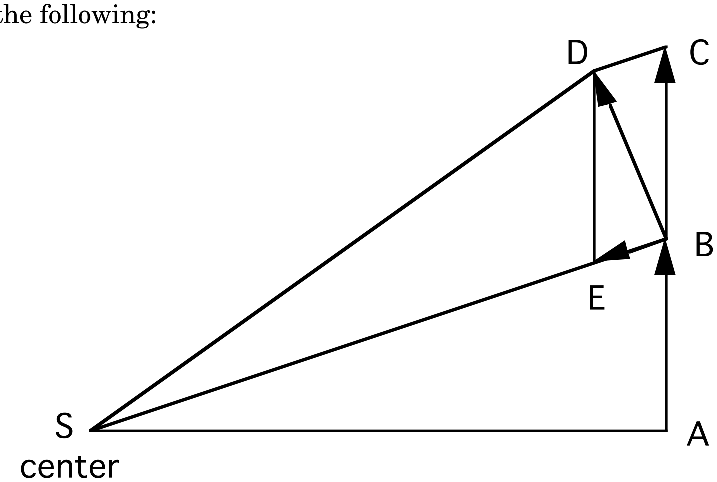
$$\sum F = ma$$

$$-G\frac{mM}{r^2}\hat{r} = m\vec{a}$$

In the equation above, the negative sign simply means that the force and acceleration of the planet is always directed towards the origin. (The unit vector $\hat{r}$ points away from the origin.) It turns out that we can generalize most of the following discussions to *any* force which depends only on a distance from the origin and is always directed to the origin: this is called a *central force*.

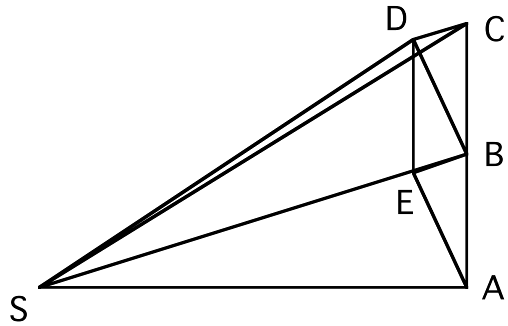
Derivation of 2nd Law *(Geometric Argument)*

After stating his 3 Laws of Motion, the first thing Newton proceeded to show was that for any central force, the area swept out per time will be a constant. (A central force is a force that is always pointed to a center, as the force of gravity on the earth is always pointed to the sun.) He used a geometrical argument similar to the following:



Imagine that a body has a certain velocity, and so would have traveled the vector **AB** in a certain amount of time. If there were no forces on the body, in the next amount of time, it would have traveled the vector **BC**. If there were a central force acting on the body, then the force would have acted towards the center shown, and the forced would have changed the velocity of the body by the vector **BE**. To find the resultant vector, we simply add the change in velocity to the original velocity, using the parallelogram BEDC, to get the resultant vector **BD**. (We could then say there would be another change in velocity, pointed along the vector **DS**, but of some other length, and draw a similar set of parallelograms to get the body going around the center.)

To Kepler from Newton



Draw in a couple more segments, SC, and EA. It can be shown that the areas of the following triangles are all equal: ΔSBD = ΔSBC = ΔSBA. Since the areas were all created in the same amount of time, then the area swept out will be the same for the same amount of time. Note that it does not matter how the force varies with the distance, all that matters is that the force always be directed to the center.

To show the areas are the same:
ΔSBC = ΔSBD because they are two triangles with the same base (SB) and the same height (BD because EBCD is a parallelogram, DC is parallel to SB.)

ΔSBD = ΔSBA because they are also two triangles with the same base (SB) and the same height. ΔBCD is congruent to ΔEBA: ∠DCB = ∠EBA because DC is parallel to EB and ABC is a line. DC = EB because EBCD is a parallelogram. CB = AB because if there was no force, the velocity would not have changed. By Side-Angle-Side, the triangles are congruent, so therefore they have the same height. The heights of the little triangles are the same as the heights of the larger triangles (height of ΔEBA is the same as the height of ΔSBA and the height of ΔABD is the same as the height of triangle ΔSBD.)

Forces do not turn on and off, but act continuously. If we imagine that the chunks of time we are considering are infinitely small, then the above diagrams become "correct."

The above derivation was adapted from the Principia. In fact, the whole Principia was written using formal geometric arguments, making it rather difficult to follow. Rather than attempt to follow Newton's logic, I will use algebra and calculus in the following derivations.

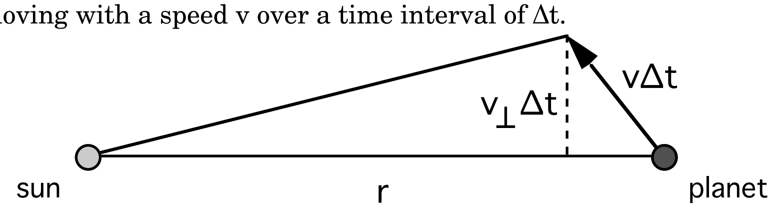
Derivation of 2nd Law *(Conservation of Angular Momentum)*

The more traditional derivation today is based on the conservation of angular momentum. For a central force acting on a body in orbit, there will be no net torque on the body, as the force will be parallel to the radius. Since the net torque is zero, the body will have a constant angular momentum. Therefore:

$$L = r \times mv$$

$$L = rmv_{\perp}$$

where r is the radius of the orbit, m is the mass of the body, and v is the velocity. The diagram below shows the planet moving with a speed v over a time interval of Δt.



The area ΔA swept out in a time Δt will be $\Delta A = \frac{1}{2} r(v_{\perp}\Delta t)$, which we can write as

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r v_{\perp}$$

To Kepler from Newton

Combining these two equations, we can say

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\left(\frac{L}{m}\right) = \frac{L}{2m}$$

Actually, the above is an approximate relationship, because the planet would be traveling in a curved path with a varying speed, not a straight line at constant speed, during the Δt. However, if we imagine making the Δt a smaller and smaller interval, so that it $\Delta t \rightarrow 0$, then we have the definition of a derivative and therefore the following is exact, and not an approximation

$$\frac{dA}{dt} = \frac{L}{2m}$$

L/2m is a constant, since the mass and angular momentum are constant, so the area per time is a constant, which is the 2nd Law. One could interpret Kepler's Second Law as a statement of conservation of Angular Momentum.

Derivation of 3rd Law

This is easy to show for the simple case of a circular orbit. A planet, mass m, orbits the sun, mass M, in a circle of radius R and a period T. The net force on the planet is a centripetal force, and is caused by the force of gravity between the sun and the planet. Therefore we can write:

$$\sum F = F_c$$

$$G\frac{mM}{R^2} = \frac{mv^2}{R}$$

The mass m of the planet cancels out. The speed of the planet going around the sun is just distance/time which is $2\pi R/T$. Substituting this makes the above equation:

$$G\frac{M}{R^2} = \frac{\left(2\pi R / T\right)^2}{R}$$

$$\frac{T^2}{R^3} = \frac{4\pi^2}{GM}$$

Note that everything on the right is a constant, so that T^2/R^3 is a constant for every planet in the solar system. This generalizes to any orbiting system. For example, if we look at T^2/R^3 for a satellite orbiting the earth and the moon, we would get the same number, and we would use the mass of the earth in the equation. (When Kepler discovered his 3rd Law, he also showed that the moons of Jupiter obeyed his Harmonic Law – and he even used that as evidence that Jupiter must rotate on its axis.)

It turns out that if we do the more formal derivation, with the two bodies orbiting about each other in ellipses, we end up with

$$\frac{T^2}{R^3} = \frac{4\pi^2}{G(m_1 + m_2)}$$

where r is the average distance between the objects, which would be the semi-major axis. Note that in the case of the planets around the sun, the mass of the sun is so much larger than the masses of any planet, the result is basically a constant for all the planets going around the sun.

To Kepler from Newton

Derivation of 1st Law *(adapted from Fowles "Analytical Mechanics, 3rd Ed.", 1977)*

Because a central force will always be directed to the origin with a magnitude that only depends on the distance away from the origin, it is very difficult to apply Newton's Laws in Cartesian coordinates. Polar coordinates are much more convenient to use because the force will only depend on *r*. In polar coordinates, Newton's Second Law for an object of mass *m* is

$$\vec{F} = m\vec{a} = m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

For a central force, the second term is zero. (By definition, a central force depends only on the distance from the origin, and is independent of angular position.) Calling the central force function f(r), we can break up the above into two equations:

1. $m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} = f(r)$
2. $m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} = 0$

In principal, given the force function, one can then solve the differential equations to find r and θ as functions of time. Since we want to find the shape of the orbit, we need to find r as a function of θ. To do this, we will change variables, letting r = 1/u.

The second equation implies that $r^2\dot{\theta}$ is a constant; which means that the angular momentum is constant. To see this,

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0$$

So the time derivative of $r^2\dot{\theta}$ is zero, which means that it is constant. If we call L the angular momentum, we can define the constant h to be

$$h = r^2\dot{\theta} = \frac{L}{m}$$

To change the variables, we need to find expressions for all the variables in equation 1.

By definition, we have

$$\text{i.}\quad r = \frac{1}{u}$$

From the constant h above we have

$$\text{ii.}\quad \dot{\theta} = \frac{h}{r^2} = hu^2$$

We need to differentiate equation i twice to find an expression for $\ddot{r}$, so

$$r = \frac{1}{u}$$

$$\dot{r} = \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

This can be rewritten as

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\left(hu^2\right)$$

So that we get

To Kepler from Newton

$$\dot{r} = -h\frac{du}{d\theta}$$

Differentiating a second time gives

$$\ddot{r} = \frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

Doing several steps in a row:

$$\begin{aligned}\ddot{r} &= -h\frac{d}{dt}\frac{du}{d\theta}\\ &= -h\frac{d}{d\theta}\frac{du}{dt}\\ &= -h\frac{d}{d\theta}\frac{du}{d\theta}\frac{d\theta}{dt}\\ &= -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}\end{aligned}$$

Substituting equation ii we get

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\left(hu^2\right)$$

So that we have

$$\text{iii.}\quad \ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Substituting these three equations into equation 1 we get

$$m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} = f(r)$$

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}\left(hu^2\right)^2\right) = f\left(\frac{1}{u}\right)$$

We finally have our generic differential equation for the orbit:

$$3.\quad \frac{du^2}{d^2\theta} + u = \frac{-f\left(1/u\right)}{mh^2u^2}$$

Equation 3 is the general differential equation of the orbit for any central force. If the central force is Newton's Law of Gravitation, then we have the following for f(r) and thus f(1/u):

$$f(r) = -G\frac{mM}{r^2}\hat{r}$$

$$f\left(\frac{1}{u}\right) = -GmMu^2$$

Note that the minus sign is because the unit radial vector $\hat{r}$ points away from the center, and the gravitational force pulls the planet into the center. Also note that *M* represents the mass of the sun, and *m* the mass of the planet.

To Kepler from Newton

Substituting this into our equation of the orbit (3) we get

$$\frac{du^2}{d^2\theta} + u = \frac{-\left(-GmMu^2\right)}{mh^2u^2} = \frac{GM}{h^2}$$

This is a relatively simple differential equation with the following as a solution

$$u = A\cos\left(\theta - \theta_0\right) + \frac{Gm}{h^2}$$

(To see that this is the solution, apply it to equation 3.) The term A is a constant of the integration, and depends on the orbital conditions of the planet. The term θ_0 is a constant of the integration, and is simply the initial orientation of the orbit. To keep things simple, lets call it 0. Changing our units back to the original gives us the following as the equation of the orbit for any body under the influence of gravity:

$$r = \frac{1}{A\cos\theta + \frac{GM}{h^2}}$$

This is the equation for a conic section in polar coordinates! We can rewrite it in the more standard form

$$r = \frac{\frac{e}{A}}{1 + e\cos\theta} \text{ with the eccentricity } e = \frac{Ah^2}{GM}$$

A is then the inverse of the distance between the focus and the directrix. Putting the equation in terms of the closest distance between the sun and the planet (the perihelion distance) r_0 gives us the following:

$$r = r_0\frac{1+e}{1 + e\cos\theta}$$

$$\text{with } e = \frac{Ah^2}{GM} \text{ and } r_0 = \frac{e}{A(1+e)} = \frac{h^2}{GM(1+e)}$$

Recalling that h = L/m, we can therefore write the following expression for perihelion, r_0:

$$r_0 = \frac{L^2}{GMm^2(1+e)}$$

For an inverse-square central force, the resulting orbit will be a conic section; the actual orbit will depend on the eccentricity as follows:

eccentricity	*resulting orbit*
$e = 0$	circle
$e < 1$	ellipse
$e = 1$	parabola
$e > 1$	hyperbola

To Kepler from Newton

Derivation of 3rd Law *(adapted from Fowles "Analytical Mechanics, 3rd Ed.", 1977)*

Here is a slightly more formal derivation of Kepler's 3rd Law – for the general case of an elliptical orbit. Recall from the 2nd Law derivation

$$\frac{dA}{dt} = \frac{L}{2m}$$

If we think about one entire orbit then t is the period of the orbit, T, and the area A would be the area of the whole orbit. (L is the angular momentum of the planet and m is the mass of the planet.) The area of an ellipse is given by πab, where a and b are the semi-major and semi-minor axis. So we could also state Kepler's 3rd Law as

$$\frac{\pi ab}{T} = \frac{L}{2m} \text{ so that } T = \frac{2\pi mab}{L}$$

The semi-minor axis, b, can be rewritten in terms of the semi-major axis, a, and the eccentricity, e, of the ellipse as $b = a\sqrt{1-e^2}$. (For an ellipse, $a^2 = b^2 + c^2$ and the eccentricity is defined as e=c/a.) In addition, we will call the semi-major axis R and rewrite the above as

$$T = \frac{2\pi m}{L}R^2\sqrt{1-e^2} \text{ which gives us } T^2 = \frac{4\pi^2 m^2}{L^2}R^4(1-e^2)$$

Notice that we can rewrite this expression as

$$T^2 = 4\pi^2 R^3\left[\frac{m^2R(1-e^2)}{L^2}\right]$$

We can see the beginnings of Kepler's 3rd Law in the expression, except we have a very cumbersome term in the brackets that we need to show is a constant, and equal to 1/GM.

In an elliptical orbit, we know $r_0 = R - c$ that and $e = \frac{c}{R}$, which we can combine as $r_0 = R - eR$ to finally get the expression

$$R = r_0\frac{1}{1-e}$$

In showing that universal gravitation leads to elliptical orbits, we derived the following expression for r_0 in terms of G, M, m, L and e:

$$r_0 = \frac{L^2}{GMm^2(1+e)}.$$

We can therefore combine these two expressions as

$$R = \left(\frac{L^2}{GMm^2(1+e)}\right)\frac{1}{1-e}$$

Simplifying and then rewriting we can therefore say

$$\frac{m^2R\left(1-e^2\right)}{L^2} = \frac{1}{GM}$$

The term on the left is exactly the cumbersome term we had found earlier. Now we can combine this expression with our earlier expression $T^2 = 4\pi^2R^3\left[\frac{m^2R(1-e^2)}{L^2}\right]$ to finally get

$$T^2 = \frac{4\pi^2}{GM}R^3$$

This is identical to our result for the derivation for the special case of the circular orbit, except that R is the semi-major axis of the elliptical orbit instead of the radius of the circular orbit.